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S-matrix at spatial infinity

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ABSTRACT

We provide a new method to construct the S-matrix in quantum field theory. This method implements crossing symmetry manifestly by erasing the a priori distinction between in- and out-states. It allows the description of processes where the interaction weakens with distance in space, but remains strong in the center at all times. It should also be applicable to certain spacetimes where the conventional method fails due to lack of temporal asymptotic states.

While the true state space of interacting quantum field theories usually is not known, the S-matrix provides an excellent tool to describe interactions using the state space of the corresponding free theory. The S-matrix is defined as a map from an initial state space of the free theory to a final one using the interaction picture where free states are invariant under free time-evolution. Its elements are limits of transition amplitudes between free states at early and at late times. The interaction is only turned on at intermediate times and the limit is determined by letting the initial and final times go to $-\infty$ and $+\infty$, respectively.

This picture is adequate if the process to be described is such that the interaction becomes negligible at very early and at very late times. It is not adequate if the interaction remains important at all times. An example of this would be a stationary bound state or, more extremely, a stationary black hole. Another shortcoming of the conventional S-matrix is that it cannot be defined for spacetimes that do not support temporal asymptotic states, such as anti-de Sitter space.

We provide here a method to construct the S-matrix that is applicable to processes where the interaction becomes negligible with distance from a center, but may remain significant there at all times. This method relies on a notion of *spatial* asymptotic states that we will explain. It is also applicable to certain spacetimes that do not support temporal asymptotic states. For example, it should allow to put the "boundary S-matrix" found for anti-de Sitter space [1] on a solid conceptual footing. However, there are also situations where the new method does not apply, but the conventional does. In particular, this is the case if space is compact (e.g., de Sitter space) or if spatial asymptotic states do not exist for some other reason.

In this Letter we present our method in Minkowski space and show that it yields the same result there as the usual one when both can be applied. This is the case when the interaction can be neglected far away from the center both in space and in time. A conceptual advantage of our method over the conventional one is that it exposes crossing symmetry as a manifest feature of the S-matrix rather than a derived one. This is due to the fact that in the present method there appears only one state space in which in- and out-states are distinguished only by their quantum numbers.

Our method is based on the general boundary formulation of quantum theory [2] and its application to quantum field theory [3,4]. The mathematical underpinning of this framework is a suitably adapted incarnation of Segal's approach to conformal field theory [5]. In short, this means we associate states to hypersurfaces in spacetime that are not restricted to be spacelike. Also, for a region in spacetime we associate an amplitude to a state that lives on its boundary. Ordinary transition amplitudes arise as special cases when the region in question is determined by a time interval. The mathematical formalism is supplemented by a physical interpretation, which allows to extract probabilities from generalized amplitudes [2,6].

Two geometries are of relevance here: (a) the spacetime region $[t_1, t_2] \times \mathbb{R}^3$ given by a time interval $[t_1, t_2]$ extended over all of space and (b) the spacetime region $\mathbb{R} \times B^3_R$, i.e., the ball of radius R in space extended over all of time. We shall refer to the latter as the *solid hypercylinder* and to its boundary $\mathbb{R} \times S^2_R$ simply as the *hypercylinder*. The first geometry appears in the standard transition amplitude between initial time t_1 and final time t_2. State spaces, amplitudes and probabilities associated with the second type of geometry where introduced in [4] for free scalar quantum field theory.

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We start by reviewing the ordinary approach to the S-matrix, deriving it as a limit of transition amplitudes between coherent states. This provides a blueprint for the subsequent section, where the asymptotic amplitude is derived that arises from coherent states on hypercylinders of increasing radius. Finally, the resulting amplitude and the conventionally derived S-matrix are shown to coincide.

For simplicity, we limit ourselves to the massive real scalar field. The generalization to other types of fields should be straightforward. We use Schrödinger (wave function) representations for states [7] and the Feynman path integral for amplitudes. We mostly follow conventions and notations of [3,4] and use the techniques laid out there.

1. Standard *S*-matrix

The following derivation of the S-matrix is similar in spirit to the method based on the holomorphic representation [8]. Standard transition amplitudes take the form

$$\langle\psi_2|U_{[t_1,t_2]}|\psi_1\rangle = N_{[t_1,t_2]}\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\psi_1(\varphi_1)\overline{\psi_2(\varphi_2)}\int_{\phi|_{t_i}=\varphi_i}\mathcal{D}\phi e^{\mathrm{i}S(\phi)}, \tag{1}$$

where $U_{[t_1,t_2]}$ is the time-evolution operator from time t_1 to time t_2. $N_{[t_1,t_2]}$ is a normalization factor such that the vacuum-to-vacuum amplitude equals one. The outer integrals are over all field configurations φ_1 and φ_2 in space. The inner integral is over all field configurations ϕ in the spacetime region $[t_1,t_2]\times\mathbb{R}^3$ subject to the boundary conditions $\phi|_{t_1}=\varphi_1$ and $\phi|_{t_2}=\varphi_2$.

We use the interaction picture to have a time independent description of free states. It will be convenient to use coherent states [9]. A coherent state ψ_η is parametrized by a complex function η on momentum space. At time t its wave function takes the form

$$\psi_{t,\eta}(\varphi)=N_{t,\eta}\exp\left(\int\frac{\mathrm{d}^3x\,\mathrm{d}^3k}{(2\pi)^3}\eta(k)e^{-\mathrm{i}(Et-kx)}\varphi(x)\right)\psi_0(\varphi), \tag{2}$$

where ψ_0 is the vacuum wave function and $N_{t,\eta}$ is a normalization factor such that the state has unit norm.

Since we are in the interaction picture, the amplitude (1) for the free action is independent of initial and final time and amounts to the S-matrix of the free theory,

$$\langle\psi_{\eta_2}|\mathcal{S}_0|\psi_{\eta_1}\rangle=\exp\left(\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E}\times\left(\eta_1(k)\overline{\eta_2(k)}-\frac{1}{2}\left|\eta_1(k)\right|^2-\frac{1}{2}\left|\eta_2(k)\right|^2\right)\right). \tag{3}$$

We now modify the free theory by adding a source field μ. That is, we take the action

$$S_\mu(\phi)=S_0(\phi)+\int\mathrm{d}^4x\,\phi(x)\mu(x), \tag{4}$$

where S_0 denotes the action of the free theory. The resulting transition amplitude again does not depend on initial and final time as long as the source field is confined to the intermediate time interval. Replacing the free action in (1) by (4) yields the S-matrix of the theory with source,

$$\langle\psi_{\eta_2}|\mathcal{S}_\mu|\psi_{\eta_1}\rangle=\langle\psi_{\eta_2}|\mathcal{S}_0|\psi_{\eta_1}\rangle\exp\left(\mathrm{i}\int\mathrm{d}^4x\,\mu(x)\hat{\eta}(x)\right)\times\exp\left(\frac{\mathrm{i}}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,\mu(x)G_F(x,x')\mu(x')\right), \tag{5}$$

where G_F is the Feynman propagator normalized such that $(\Box_x+M^2)G_F(x,x')=\delta^4(x-x')$. $\hat{\eta}$ is the complex solution of the Klein–Gordon equation given by

$$\hat{\eta}(t,x)=\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left(\eta_1(k)e^{-\mathrm{i}(Et-kx)}+\overline{\eta_2(k)}e^{\mathrm{i}(Et-kx)}\right). \tag{6}$$

We note that the initial and final coherent states determine the positive and negative energy contributions to the solution. Conversely, (6) allows us to recover a pair of coherent states from a complex solution.

This result combined with functional methods can be used to work out the S-matrix for the general interacting theory. Consider the action

$$\begin{aligned}S(\phi)&=S_0(\phi)+\int\mathrm{d}^4x\,V\big(x,\phi(x)\big)\\&=S_0(\phi)+\int\mathrm{d}^4x\,V\left(x,\frac{\partial}{\partial\mu(x)}\right)S_\mu(\phi)\bigg|_{\mu=0}.\end{aligned} \tag{7}$$

We assume at first that the interaction is cut off at early and at late times, i.e., $V((t,x),a)=0$ if $t\leqslant t_1$ or $t\geqslant t_2$. The transition amplitude (1) with (7) inserted then yields

$$\langle\psi_{\eta_2}|\mathcal{S}|\psi_{\eta_1}\rangle=\exp\left(\mathrm{i}\int\mathrm{d}^4x\,V\left(x,-\mathrm{i}\frac{\partial}{\partial\mu(x)}\right)\right)\times\langle\psi_{\eta_2}|\mathcal{S}_\mu|\psi_{\eta_1}\rangle\bigg|_{\mu=0}. \tag{8}$$

Again, there is no explicit dependence on t_1 or t_2 so we can drop the cutoff and interpret the result as the S-matrix of the interacting theory.

2. *S*-matrix from timelike hypercylinders

In this section we make heavy use of the results of [4]. The amplitude associated with the solid hypercylinder of radius R for a state ψ takes the form,

$$\rho_R(\psi)=N_R\int\mathcal{D}\varphi\psi_R(\varphi)\int_{\phi|_R=\varphi}\mathcal{D}\phi e^{\mathrm{i}S(\phi)}, \tag{9}$$

where N_R is a normalization factor, such that the amplitude of the vacuum state equals one. The outer integral is over field configurations φ on the hypercylinder of radius R. The inner integral is over field configurations ϕ in the interior $\mathbb{R}\times B^3_R$ of the hypercylinder matching φ on the boundary.

The interaction picture is now defined by describing free states in a radius-independent form. That is, we identify states which are related by radial evolution. (To do this one needs the amplitude associated to a hypertube [4].) Again we use coherent states. A coherent state ψ_ξ in this setting may be characterized by a set of functions $\xi_{l,m}(E)$ that carry angular momentum quantum numbers l,m and depend on the energy E. The energy may be positive or negative, but $\xi_{l,m}(E)=0$ if $|E|<M$. The latter condition comes from the fact that the particle spectrum on the hypercylinder is confined to non-negative values of $p^2=E^2-M^2$ [4]. The wave function of the coherent state at radius R takes the form

$$\psi_{R,\xi}(\varphi)=N_{R,\xi}\exp\left(\int\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}E\sum_{l,m}\times\xi_{l,m}(E)\frac{e^{\mathrm{i}Et}Y_l^{-m}(\Omega)}{2\pi h_l(pR)}\varphi(t,\Omega)\right)\psi_{R,0}(\varphi), \tag{10}$$

where $\psi_{R,0}$ is the vacuum wave function at radius R and $N_{R,\xi}$ is a normalization factor such that the state has unit norm. Here, Y_l^m denotes the spherical harmonic and h_l the spherical Bessel function of the third kind. The inner product of coherent states is then

$$\langle\psi_{R,\xi'},\psi_{R,\xi}\rangle=\exp\biggl(\int \mathrm{d}E\sum_{l,m}\frac{p}{4\pi}\biggl(\xi_{l,m}(E)\overline{\xi'_{l,m}(E)} -\frac{1}{2}\bigl|\xi_{l,m}(E)\bigr|^2-\frac{1}{2}\bigl|\xi'_{l,m}(E)\bigr|^2\biggr)\biggr). \tag{11}$$

Since we use the interaction picture, the amplitude of a state is independent of the radius R,

$$\mathcal{S}_0(\psi_\xi)=\rho_{R,0}(\psi_{R,\xi})=\exp\biggl(\int \mathrm{d}E\sum_{l,m}\frac{p}{8\pi} \times\bigl(\xi_{l,m}(E)\xi_{l,-m}(-E)-\bigl|\xi_{l,m}(E)\bigr|^2\bigr)\biggr). \tag{12}$$

We turn to the theory with source field μ given by the action (4). Working out the corresponding path integral (9) when the source is confined to the interior of the hypercylinder of radius R yields,

$$\mathcal{S}_\mu(\psi_\xi)=\mathcal{S}_0(\psi_\xi)\exp\biggl(\mathrm{i}\int \mathrm{d}^4x\,\mu(x)\hat{\xi}(x)\biggr) \times\exp\biggl(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,\mu(x)G_F(x,x')\mu(x')\biggr), \tag{13}$$

where G_F is the Feynman propagator. $\hat{\xi}$ is the complex solution of the Klein–Gordon equation given by

$$\hat{\xi}(t,r,\Omega)=\int \mathrm{d}E\sum_{l,m}\frac{p}{2\pi}\xi_{l,m}(E)j_l(pr)e^{\mathrm{i}Et}Y_l^{-m}(\Omega). \tag{14}$$

Here, j_l denotes the spherical Bessel function of the first kind. We note that this equation puts into correspondence coherent states with complex solutions. Since (13) does not depend on the radius R, the restriction on μ to vanish outside of R may be lifted.

As before, we use functional methods to work out the asymptotic amplitude for the general interacting theory. Take the action (7) and assume at first that the interaction is cut off outside the radius R, i.e., $V((t,x),a)=0$ if $|x|\geqslant R$. The amplitude (9) with (7) inserted then yields

$$\mathcal{S}(\psi_\xi)=\exp\biggl(\mathrm{i}\int \mathrm{d}^4x\,V\biggl(x,-\mathrm{i}\frac{\partial}{\partial\mu(x)}\biggr)\biggr)\mathcal{S}_\mu(\psi_\xi)\bigg|_{\mu=0}. \tag{15}$$

Again, there is no explicit dependence on R so we can drop the cutoff. Then (15) is the asymptotic amplitude of the interacting theory.

3. Equivalence of states and asymptotic amplitudes

It is striking how much the expressions for the usual S-matrix and the spatially asymptotic amplitude resemble each other. It should be emphasized that this is a priori not at all obvious. In particular, the fact that both in (5) and in (13) the same Feynman propagator appears is rather non-trivial.

Let us explain this a little. The exponential term containing the Feynman propagator in (5) arises in the form

$$\exp\biggl(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mu(x)\alpha(x)\biggr), \tag{16}$$

where α is a complex solution of the inhomogeneous Klein–Gordon equation $(\Box+M^2)\alpha=\mu$. α must satisfy boundary conditions at early and at late time. Expanding α in terms of plane waves as

$$\alpha(t,x)=\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E}\bigl(\alpha^+(t,k)e^{-\mathrm{i}(Et-kx)}+\alpha^-(t,k)e^{\mathrm{i}(Et-kx)}\bigr), \tag{17}$$

the boundary conditions are $\alpha^+(t,k)=0$ for early times t before the source is switched on and $\alpha^-(t,k)=0$ for late times t after the source is switched off. These are just the Feynman boundary condition leading to the substitution $\alpha(x)=\int \mathrm{d}^4x' G_F(x,x')\mu(x')$.

Similarly, in the hypercylinder setting the exponential containing the Feynman propagator in (13) also arises from an expression of the form (16). Again α is a complex solution of the inhomogeneous Klein–Gordon equation. However, this time the boundary conditions for α arise at large radius rather than at large (early or late) time. Expanding α in terms of plane waves in time and spherical harmonics in space,

$$\alpha(t,r,\Omega)=\int \mathrm{d}E\sum_{l,m}\bigl(\alpha^<_{l,m}(r,E)h_l(pr)e^{-\mathrm{i}Et}Y_l^m(\Omega) +\alpha^>_{l,m}(r,E)\bar{h}_l(pr)e^{\mathrm{i}Et}Y_l^{-m}(\Omega)\bigr), \tag{18}$$

the boundary condition is $\alpha^>_{l,m}(r,E)=0$ for large radius r outside of the source field μ. Surprisingly, this boundary condition turns out to be equivalent to the Feynman boundary condition, yielding the same propagator. We shall elaborate more on this elsewhere [10].

If states and amplitudes associated with hypercylinders yield as valid a description of physics as the usual one we should be able to compare the descriptions and even conclude their equivalence. Indeed, consider a process (represented by an interaction) that is bounded in space and in time. It should be possible to describe it either via usual transition amplitudes for sufficiently early initial time and late final time, or through amplitudes for a solid hypercylinder of sufficiently large radius. What is more, when we let the region where the process takes place grow arbitrarily large (in space and in time) we should still get equivalent results. This is indeed the case.

To compare the two settings we need a map between the different boundary state spaces. In the standard setting the boundary state space associated with the interval $[t_1,t_2]$ is the tensor product $\mathcal{H}_{t_1}\otimes\mathcal{H}^*_{t_2}$. (The dual at t_2 is related to the fact that we should think of final states as bra-states rather than ket-states.) We denote the state space associated with the hypercylinder of radius R by $\mathcal{H}_R$. Thus, we are looking for an isomorphism of Hilbert spaces $\mathcal{H}_{t_1}\otimes\mathcal{H}^*_{t_2}\cong\mathcal{H}_R$. What is more, the relevant (asymptotic) amplitudes should be equal under this isomorphism. Comparing (5) with (13) shows that this can be true only if the isomorphism is given as follows: $\psi_{\eta_1}\otimes\psi_{\eta_2}\cong\psi_\xi$ if and only if $\hat{\eta}=\hat{\xi}$. As we have remarked before, (6) and (14) really establish bijective correspondences between classical solutions and coherent states. Hence, the proposed isomorphism is really bijective. Moreover, it is indeed an isomorphism (i.e., preserves the inner product). One also easily checks that the free amplitudes (3) and (12) are equal under the isomorphism. The same follows for the amplitudes with source (5) and (13). Finally, it is then obvious that the asymptotic amplitudes with general interaction (8) and (15) also coincide.

Using the definition of particle states on the hypercylinder from [4], it is easy to verify that the isomorphism of state spaces constructed above indeed sends an in-state with m particles and an out-state with n particles to a states with $m+n$ particles on the hypercylinder. What is more, the latter have quantum numbers (the sign of the energy) that identify them correctly as in- or out-going. However, the meaning of in- or out-going is then with respect to the region defined by the solid hypercylinder. This reinforces the notion that in- versus out-going is not primarily a temporal property, but a spatiotemporal one [3,4]. It indicates whether a particle goes into or comes out of the (interaction) region of interest.

It is also clear how crossing symmetry in the hypercylinder case is simply implicit in the fact that all particles are part of a single state space. One could then view the isomorphism of state spaces as yielding a derivation of crossing symmetry of the S-matrix in the standard setup.

It remains to remark on the probabilities extracted from the S-matrix. In [4] it was shown how the generalized probability interpretation introduced in [2] applies to amplitudes associated with the solid hypercylinder. Using the methods of [4] one can then show that the usual probability interpretation of the S-matrix arises as a special case.

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